

Middle East Technical University
Department of Mechanical Engineering
ME 413 Introduction to Finite Element Analysis

Chapter 4
Trusses, Beams and Frames

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These notes are prepared with the hope to be useful to those who want to learn and teach FEM. You are free to use them. Please send feedbacks to the above email address.

What Is This Chapter About?

- We'll study FEM formulations of
 - deformation of planar **trusses**
 - bending of **beams**
 - deformation of **frames** (as the superposition of planar truss and beam formulations)
- These problems will be studied as **1D**, but there will be multiple unknowns at a node.
- We'll modify the **1D FEM code** to solve these problems.



Deformation of a Bar

- A **bar** is a structural member that is loaded axially.



- It is either in direct tension or compression.
- Axial deformation, u , is governed by the following DE

$$-\frac{d}{dx} \left(EA \frac{du}{dx} \right) = 0$$

solution of which is **linear for constant E and A** .

- Even a single linear element can solve this problem exactly.

Deformation of a Bar (cont'd)

- Elemental weak form of the problem is

$$\int_{\Omega^e} EA \frac{du}{dx} \frac{dw}{dx} dx = \underbrace{\left[wEA \frac{du}{dx} \right]_{x_2^e}}_{Q_2^e} + \underbrace{\left[-wEA \frac{du}{dx} \right]_{x_1^e}}_{Q_1^e}$$

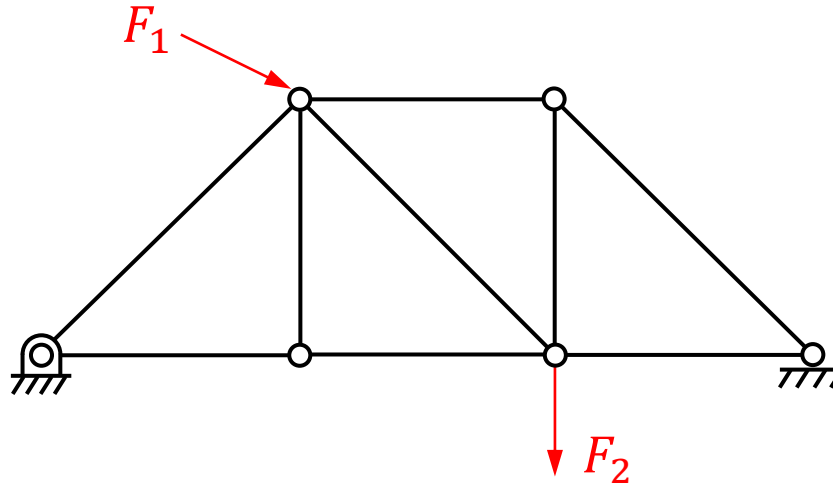
- SV of the problem is the **axial force** : $EA \frac{du}{dx} n_x$

- Elemental stiffness matrix is $K_{ij}^e = \int_{\Omega^e} EA \frac{dS_i}{d\xi} \frac{1}{J^e} \frac{dS_j}{d\xi} \frac{1}{J^e} J^e d\xi$

- Elemental system is $\frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$
↙ Elemental force vector is zero

Planar Truss

- A truss consists of several bars connected with frictionless pin joints.
- Note that this is not the actual meaning of truss in civil engineering.

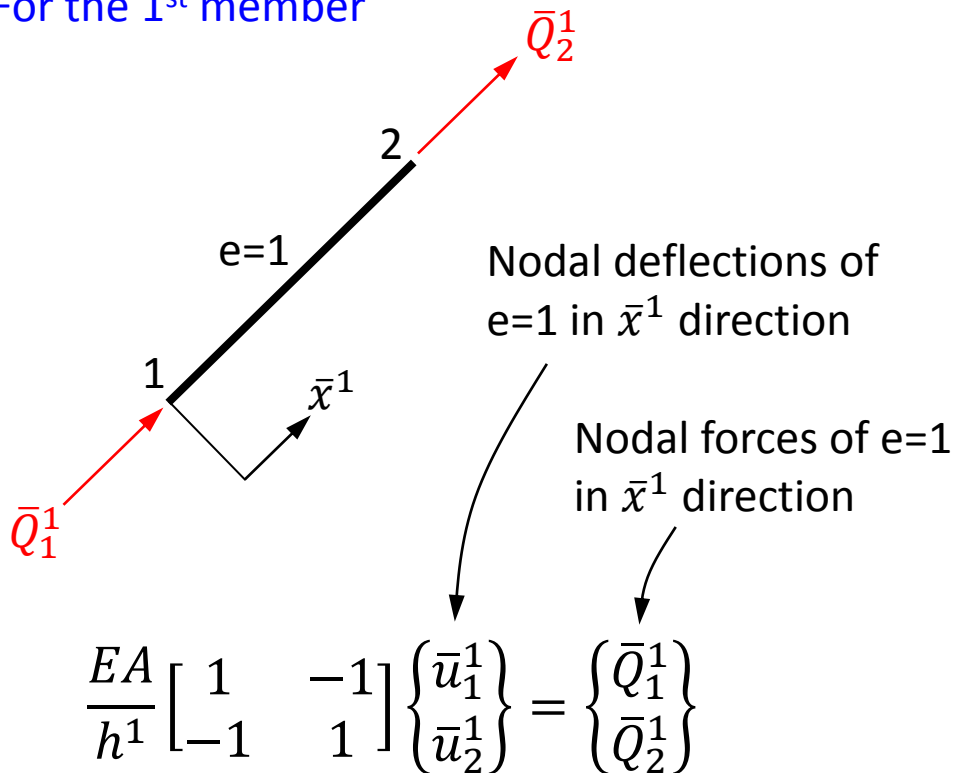


- Each member can **only carry axial force**, but no shear force or bending moment.
- All members of a **planar truss** lie on a 2D plane. **Space truss** is the 3D version.
- A truss can be loaded with multiple **point forces** at its joints.
- Typically there is at least one **fixed joint**.
- Some joints might have **restricted motion**.
- **Deformations are small**, i.e. general shape of the truss is similar before and after loading.

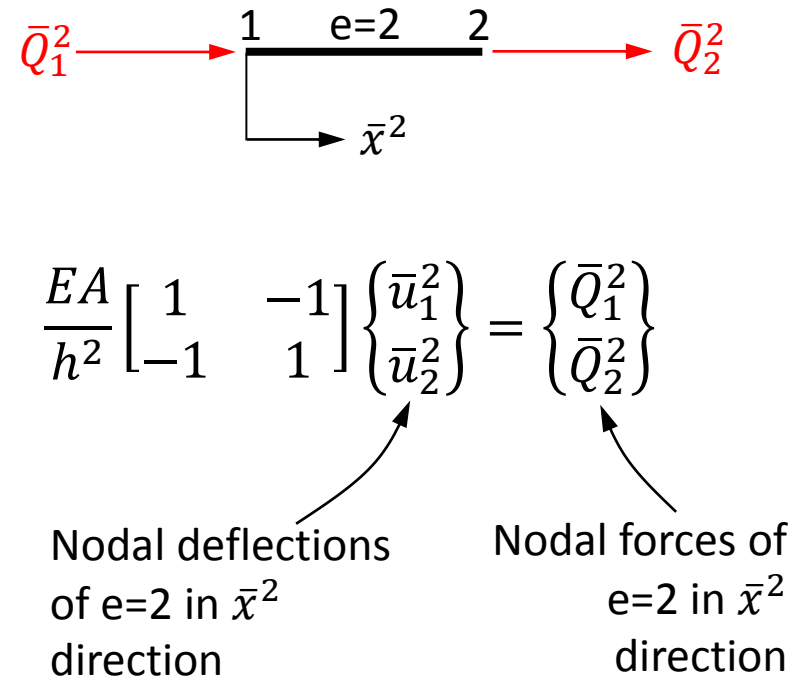
Planar Truss – Local Coordinates

- Each member of a truss can be treated as an element of a FE mesh.
- The elemental system derived previously for a bar is valid for each member.
- But in order to be able to use it, different coordinate systems aligned with each member should be used. These **local coordinates** are shown below with \bar{x}^1 and \bar{x}^2 .

For the 1st member

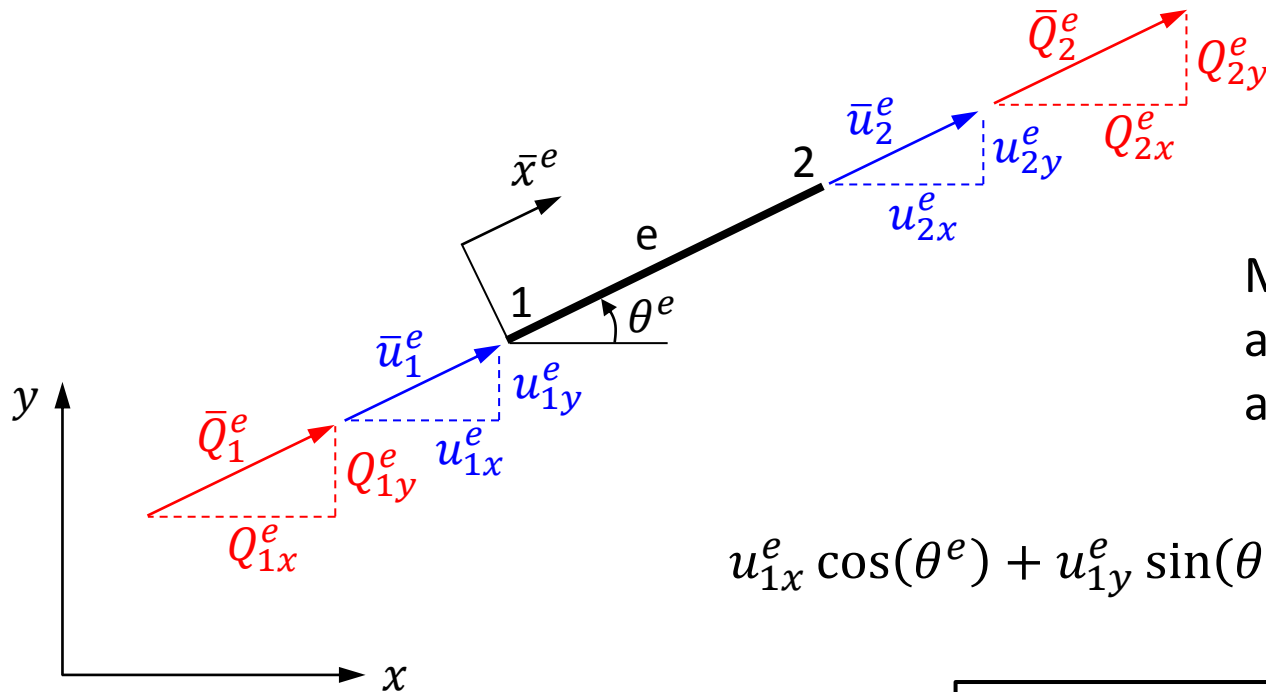


For the 2nd member



Planar Truss – Transformation Matrix

- During the assembly of the elemental systems, PVs and SVs written for a common xy coordinate system should be used.
- For each element a transformation between local \bar{x} coordinate and the global xy coordinates is necessary.
- This is a **purely geometrical transformation**.



$$u_{1x}^e = \bar{u}_1^e \cos(\theta^e)$$

$$u_{1y}^e = \bar{u}_1^e \sin(\theta^e)$$

Multiply the 1st eqn with $\cos(\theta^e)$ and the 2nd eqn with $\sin(\theta^e)$ and add them up.

$$u_{1x}^e \cos(\theta^e) + u_{1y}^e \sin(\theta^e) = \bar{u}_1^e \underbrace{[\cos^2(\theta^e) + \sin^2(\theta^e)]}_1$$

$$\bar{u}_1^e = u_{1x}^e \cos(\theta^e) + u_{1y}^e \sin(\theta^e)$$

Planar Truss – Transformation Matrix (cont'd)

- Similar for the 2nd node of element e :
- Together these two eqns become

$$\bar{u}_2^e = u_{2x}^e \cos(\theta^e) + u_{2y}^e \sin(\theta^e)$$

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta^e) & \sin(\theta^e) & 0 & 0 \\ 0 & 0 & \cos(\theta^e) & \sin(\theta^e) \end{bmatrix}}_{\text{Transformation matrix, } [T^e]} \begin{Bmatrix} u_{1x}^e \\ u_{1y}^e \\ u_{2x}^e \\ u_{2y}^e \end{Bmatrix}$$

$$\{\bar{u}^e\} = [T^e]\{\Delta^e\}$$

Δ^e includes both u_x^e 's and u_y^e 's.
Each node has 2 unknowns and in total one element has 4 unknowns.

- A similar eqn can be written for the SVs too

$$\{\bar{Q}^e\} = [T^e]\{Q^e\}$$

$$\{Q^e\} = \{Q_{1x}^e \quad Q_{1y}^e \quad Q_{2x}^e \quad Q_{2y}^e\}^T$$

Planar Truss – $[K^e]$ Transformation Matrix

- $[T^e]$ can be used to transform the original 2x2 elemental system into a new 4x4 elemental system

- Original 2x2 elemental system using bars :
$$\frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{Bmatrix} = \begin{Bmatrix} \bar{Q}_1^e \\ \bar{Q}_2^e \end{Bmatrix}$$

or

$$[\bar{K}^e] \{\bar{u}^e\} = \{\bar{Q}^e\}$$

- Using $\{\bar{u}^e\} = [T^e] \{\Delta^e\}$ and $\{\bar{Q}^e\} = [T^e] \{Q^e\}$

$$[\bar{K}^e] [T^e] \{\Delta^e\} = [T^e] \{Q^e\}$$

- Premultiply this eqn by $[T^e]^T$

$$\underbrace{[T^e]^T [\bar{K}^e] [T^e]}_{[K^e]} \{\Delta^e\} = \underbrace{[T^e]^T [T^e]}_{[I] : \text{Identity matrix}} \{Q^e\}$$

Planar Truss – Transformed $[K^e]$

- Transformed 4x4 elemental system : $[K^e]\{\Delta^e\} = \{Q^e\}$

$$[T^e]^T [\bar{K}^e] [T^e]$$

$$\begin{bmatrix} \alpha & 0 \\ \beta & 0 \\ 0 & \alpha \\ 0 & \beta \end{bmatrix} \frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta & 0 & 0 \\ 0 & 0 & \alpha & \beta \end{bmatrix}$$

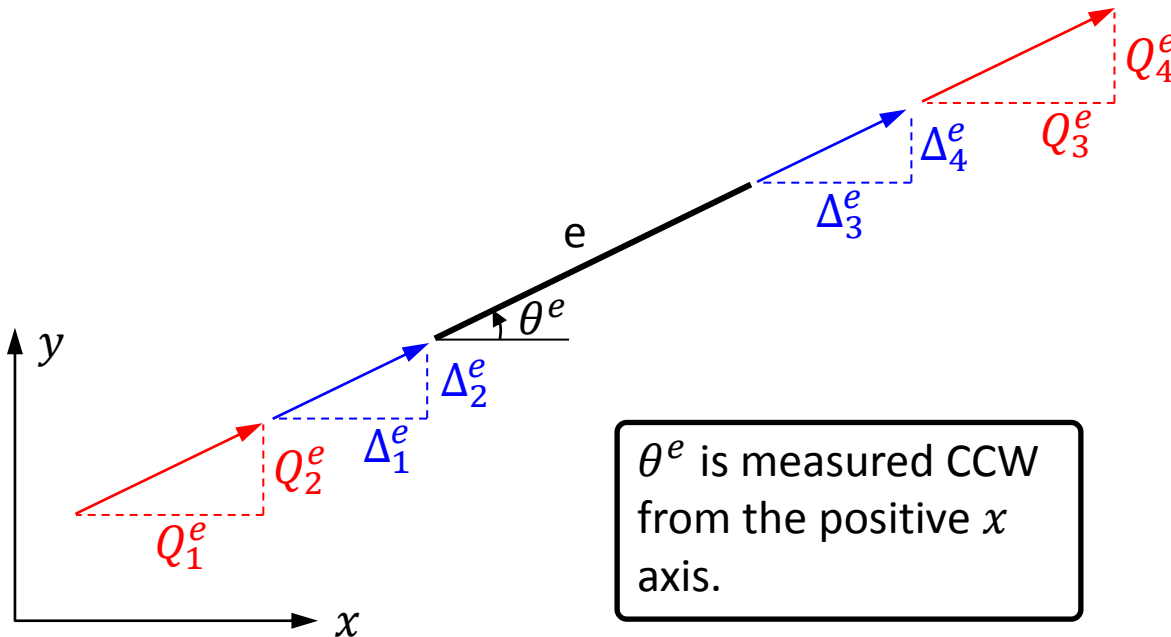
$$[K^e] = \frac{EA}{h^e} \begin{bmatrix} \alpha^2 & \alpha\beta & -\alpha^2 & -\alpha\beta \\ & \beta^2 & -\alpha\beta & -\beta^2 \\ & & \alpha^2 & \alpha\beta \\ \text{sym} & & & \beta^2 \end{bmatrix}$$

where $\alpha = \cos(\theta^e)$ and $\beta = \sin(\theta^e)$

Planar Truss – 4x4 Elemental System

- Therefore in a planar truss solution elemental systems are 4x4.
- Each truss member is a linear element with 4 unknowns

$$\frac{EA}{h^e} \begin{bmatrix} \alpha^2 & \alpha\beta & -\alpha^2 & -\alpha\beta \\ & \beta^2 & -\alpha\beta & -\beta^2 \\ & & \alpha^2 & \alpha\beta \\ \text{sym} & & & \beta^2 \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$



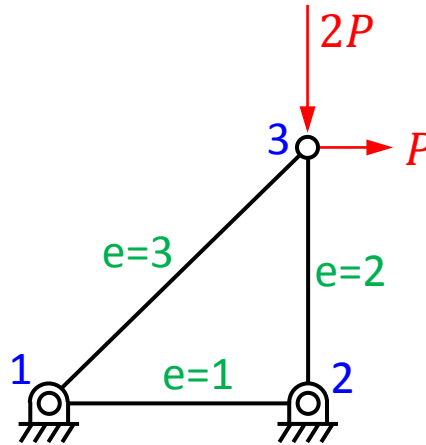
θ^e is measured CCW from the positive x axis.

Δ_1^e : Horizontal deflection of point 1
 Δ_2^e : Vertical deflection of point 1
 Δ_3^e : Horizontal deflection of point 2
 Δ_4^e : Vertical deflection of point 2

Q_1^e : Horizontal force at point 1
 Q_2^e : Vertical force at point 1
 Q_3^e : Horizontal force at point 2
 Q_4^e : Vertical force at point 2

Planar Truss – Local to Global Unknown Mapping

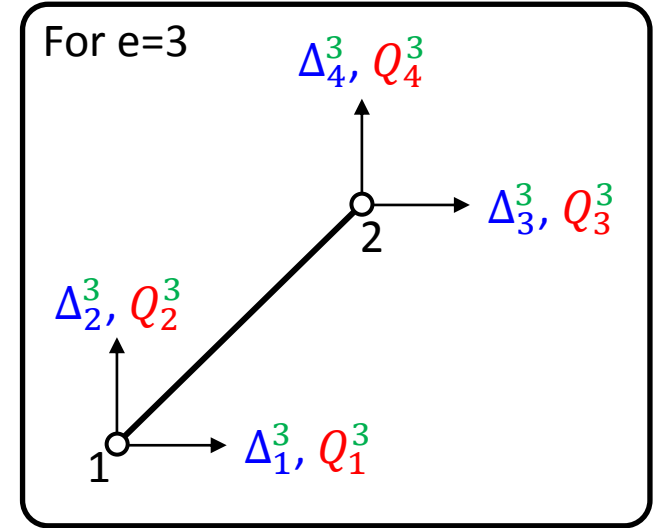
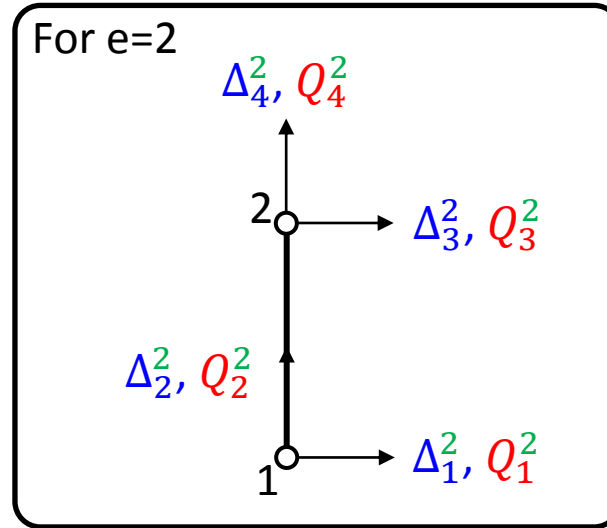
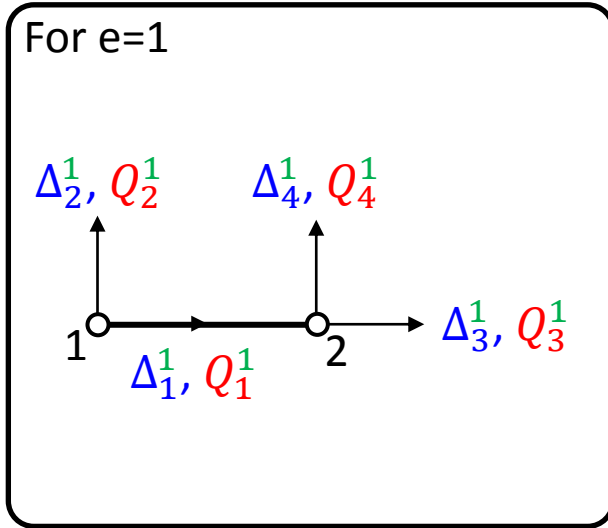
- Consider the following truss problem.



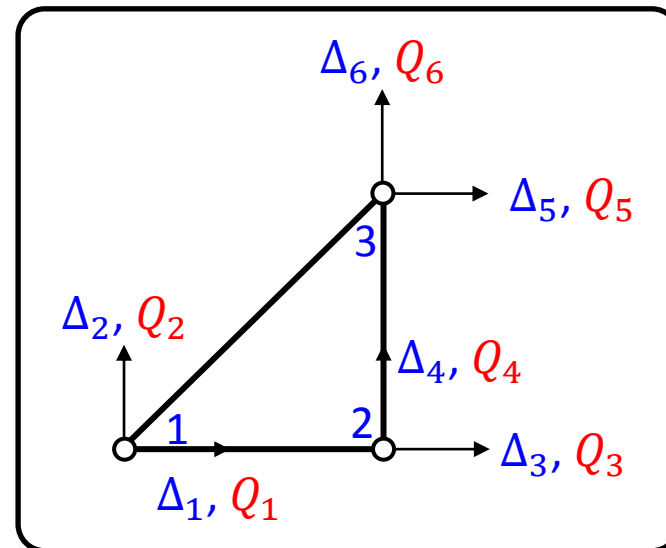
- There are $NE = 3$ elements and $NN = 3$ nodes.
- At each node there are $NNU = 2$ unknown deflections. Totally there are $NU = 6$ unknowns.
- 4 of these unknowns are known. Nodes 1 and 2 are fixed.
- We need to determine 2 deflections (horizontal and vertical) at node 3 and, if desired, the reaction forces at nodes 1 and 2.

Planar Truss – Local to Global Unknown Mapping (cont'd)

- PVs and SVs of each element are

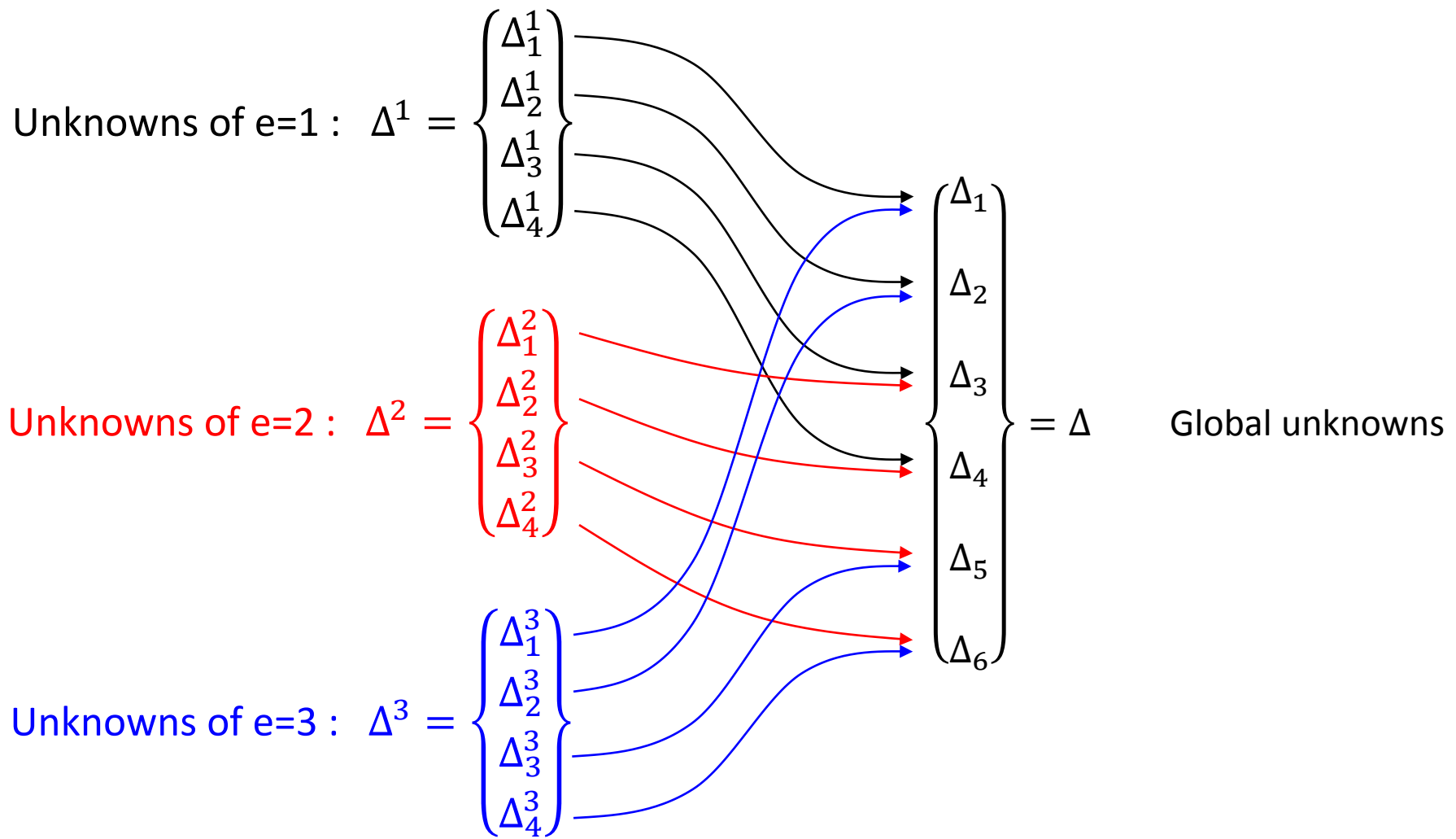


- PVs and SVs of the global system are
- In general the global PVs (and SVs) of the i^{th} node are numbered as $2i - 1$ and $2i$.



Planar Truss – Local to Global Unknown Mapping (cont'd)

- Assembly process is about **local-to-global unknown mapping** for each element



Planar Truss – $LtoG$ Matrix & Assembly Rule

- This graph can also be expressed as a **local-to-global mapping matrix**

$$LtoG = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 \end{bmatrix} \begin{array}{l} \leftarrow e=1 \\ \leftarrow e=2 \\ \leftarrow e=3 \end{array}$$

- $LtoG_{ij}$ gives the global unknown number of the i^{th} element's j^{th} local unknown.
- For example, $LtoG_{34} = 6$ because the 3rd element's 4th local unknown is the 6th global unknown.

The **assembly rule** can now be defined as

- K_{ij}^e entry of an elemental system goes to K_{IJ} entry of the global system.
- F_i^e entry of an elemental system goes to F_I entry of the global system.
- Q_i^e entry of an elemental system goes to Q_I entry of the global system.

where $I = LtoG_{ei}$
 $J = LtoG_{ej}$

Planar Truss – Assembly

- Using the assembly rule assemble system of the 3-member truss is

$$\begin{bmatrix}
 K_{11}^1 + K_{11}^3 & K_{12}^1 + K_{12}^3 & K_{13}^1 & K_{14}^1 & K_{13}^3 & K_{14}^3 \\
 K_{21}^1 + K_{21}^3 & K_{22}^1 + K_{22}^3 & K_{23}^1 & K_{24}^1 & K_{23}^3 & K_{24}^3 \\
 K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 \\
 K_{41}^1 & K_{42}^1 & K_{43}^1 + K_{21}^2 & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^2 \\
 K_{31}^3 & K_{32}^3 & K_{31}^2 & K_{32}^2 & K_{33}^2 + K_{33}^3 & K_{34}^2 + K_{34}^3 \\
 K_{41}^3 & K_{42}^3 & K_{41}^2 & K_{42}^2 & K_{43}^2 + K_{43}^3 & K_{44}^2 + K_{44}^3
 \end{bmatrix}
 \begin{Bmatrix}
 \Delta_1 \\
 \Delta_2 \\
 \Delta_3 \\
 \Delta_4 \\
 \Delta_5 \\
 \Delta_6
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 Q_1^1 + Q_1^3 \\
 Q_2^1 + Q_2^3 \\
 Q_3^1 + Q_1^2 \\
 Q_4^1 + Q_2^2 \\
 Q_3^2 + Q_3^3 \\
 Q_4^2 + Q_4^3
 \end{Bmatrix}$$

which only depends on the *LtoG* matrix

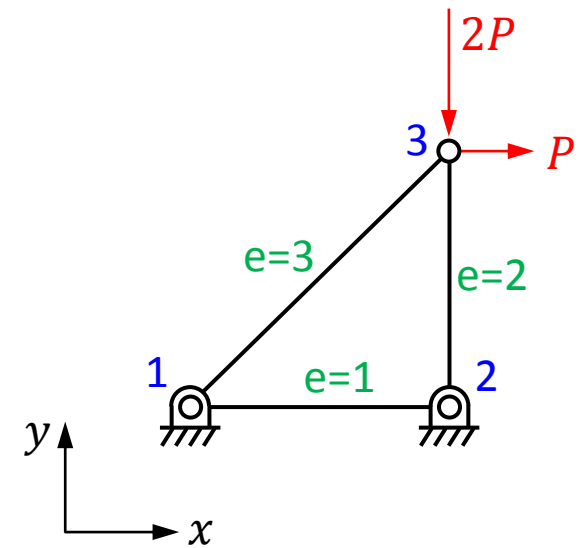
$$LtoG = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 \end{bmatrix}$$

- LtoG* matrix depends on
 - how we number the nodes globally
 - how we number the elements' nodes locally, i.e. which node is the 1st and which one is the 2nd ?

Planar Truss – Point Loads

- Consider the following truss problem with 2 point loads.
- How should we use the points loads?
- They are used in the boundary term vector.
- $\{Q\}$ of this problem is

$$\{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^2 + Q_4^3 \end{Bmatrix}$$

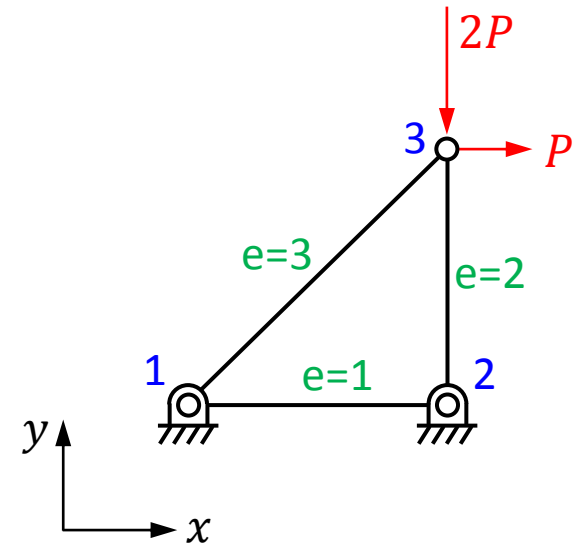


- If there is no horizontal (or vertical) force at a node, the corresponding SV is set to zero.
- If there is a given point load at a node, the corresponding SV is set to the given value.
- At supports SV(s) are unknown and can be calculated during post-processing.
- Be careful with the direction (sign) of forces.

Planar Truss – Point Loads

- For our problem
- Q_1 and Q_2 are unknown reaction forces at node 1.
- Q_3 and Q_4 are unknown reaction forces at node 2.
- $Q_5 = P$ (given horizontal point load at node 3).
- $Q_6 = -2P$ (given vertical point load at node 3).

- Therefore the $\{Q\}$ vector is
$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ P \\ -2P \end{Bmatrix}$$



- Q_1, Q_2, Q_3 and Q_4 are not known, but the corresponding $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 are known.
- If there were no horizontal load at node 3, we should have set Q_5 to zero.
- If there were no vertical load at node 3, we should have set Q_6 to zero.
- If there were a roller support that restricts vertical motion but not horizontal motion, we should have set Q_4 to zero.

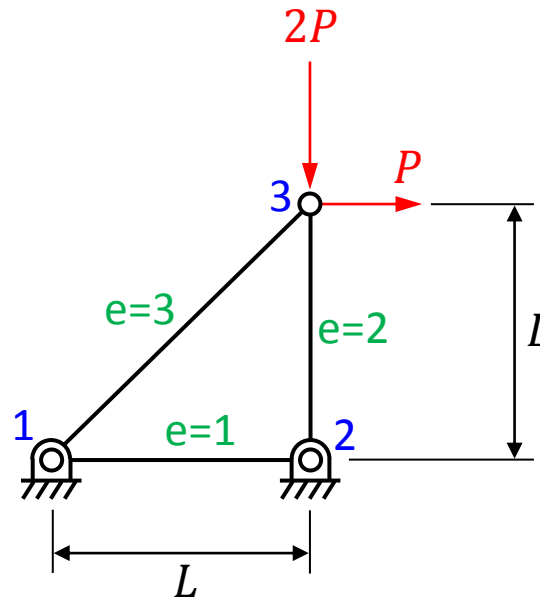
Example 4.1



Example 4.1 : Solve the following truss problem.

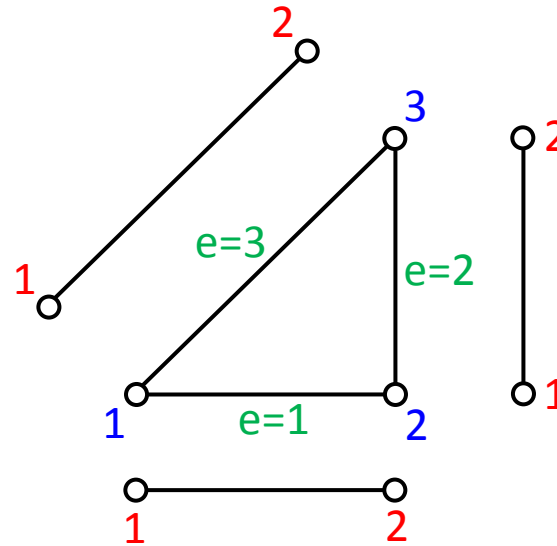
- Find the deflection of the nodes.
- Determine the forces and stresses in each member.
- Determine the reaction forces at the supports.

E and A values are the same for each member.



Example 4.1 (cont'd)

- Elemental system equation is given in slide 2-10.
- θ values are necessary and they depend on local node numbering of the nodes



- θ values of the elements are : $\theta^1 = 0$, $\theta^2 = \pi/2$, $\theta^3 = \pi/4$
- **Note :** If the local node numbering of all the elements are reversed, θ values change as $\theta^1 = \pi$, $\theta^2 = 3\pi/2$, $\theta^3 = 5\pi/4$

Example 4.1 (cont'd)

- Elemental systems are

For e=1:

$$\begin{aligned}\alpha &= \cos(\theta^1) = 1 \\ \beta &= \sin(\theta^1) = 0 \\ h^1 &= L\end{aligned} \quad \rightarrow \quad [K^1] = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & \text{sym.} & 1 & 0 \\ & & & 0 \end{bmatrix}$$

For e=2:

$$\begin{aligned}\alpha &= \cos(\theta^2) = 0 \\ \beta &= \sin(\theta^2) = 1 \\ h^2 &= L\end{aligned} \quad \rightarrow \quad [K^2] = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & \text{sym.} & 0 & 0 \\ & & & 1 \end{bmatrix}$$

For e=3:

$$\begin{aligned}\alpha &= \cos(\theta^3) = 1/\sqrt{2} \\ \beta &= \sin(\theta^3) = 1/\sqrt{2} \\ h^3 &= \sqrt{2}L\end{aligned} \quad \rightarrow \quad [K^3] = \frac{EA}{2\sqrt{2}L} \begin{bmatrix} 1 & 1 & -1 & -1 \\ & 1 & -1 & -1 \\ & \text{sym.} & 1 & 1 \\ & & & 1 \end{bmatrix}$$

Example 4.1 (cont'd)

- Assembled system is given in slide 4-16. With numbers it becomes

$$\frac{EA}{L} \begin{bmatrix} 1 + 0.3536 & 0 + 0.3536 & -1 & 0 & -0.3536 & -0.3536 \\ & 0 + 0.3536 & 0 & 0 & -0.3536 & -0.3536 \\ & & 1 + 0 & 0 + 0 & 0 & 0 \\ & & & 0 + 1 & 0 & -1 \\ & \text{symmetric} & & & 0 + 0.3536 & 0 + 0.3536 \\ & & & & & 1 + 0.3536 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ P \\ -2P \end{Bmatrix}$$

- We know that

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0$$

- Reduction can be applied to get the following 2x2 system

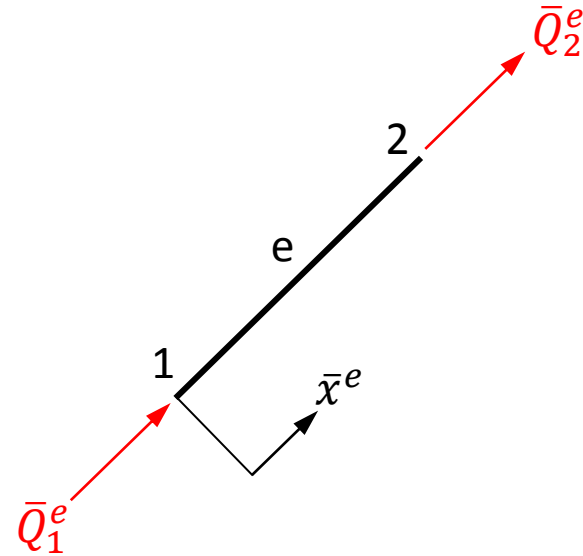
$$\begin{bmatrix} 0.3536 & 0.3536 \\ 0.3536 & 1.3536 \end{bmatrix} \begin{Bmatrix} \Delta_5 \\ \Delta_6 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix}$$

- Solving this we get

$$\Delta_5 = 5.828 \frac{PL}{EA} \quad , \quad \Delta_6 = -3 \frac{PL}{EA}$$

Example 4.1 (cont'd)

- To calculate **axial forces** in each member we can go back to local coordinates aligned with the elements.



- From slide 4-9

$$\begin{Bmatrix} \bar{Q}_1^e \\ \bar{Q}_2^e \end{Bmatrix} = \frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{Bmatrix}$$

- Using the transformation matrix definition from Slide 4-8

$$\begin{Bmatrix} \bar{Q}_1^e \\ \bar{Q}_2^e \end{Bmatrix} = \frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} [T^e] \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix}$$

Example 4.1 (cont'd)

- **Axial forces** in each member are

- For e=1 :
$$\begin{Bmatrix} \bar{Q}_1^1 \\ \bar{Q}_2^1 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- For e=2 :
$$\begin{Bmatrix} \bar{Q}_1^2 \\ \bar{Q}_2^2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 5.828 \frac{PL}{EA} \\ -3 \frac{PL}{EA} \end{Bmatrix} = \begin{Bmatrix} 3P \\ -3P \end{Bmatrix}$$

- For e=3 :
$$\begin{Bmatrix} \bar{Q}_1^3 \\ \bar{Q}_2^3 \end{Bmatrix} = \frac{EA}{\sqrt{2}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 5.828 \frac{PL}{EA} \\ -3 \frac{PL}{EA} \end{Bmatrix} = \begin{Bmatrix} -1.414P \\ 1.414P \end{Bmatrix}$$

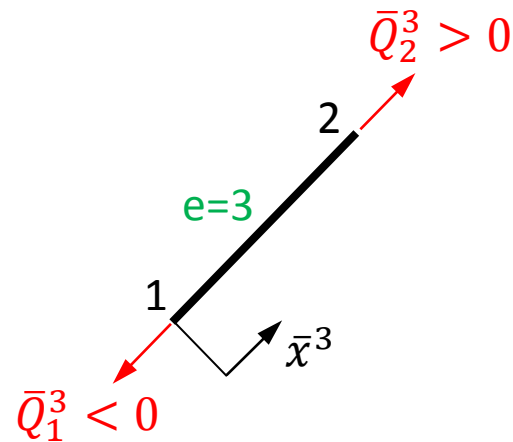
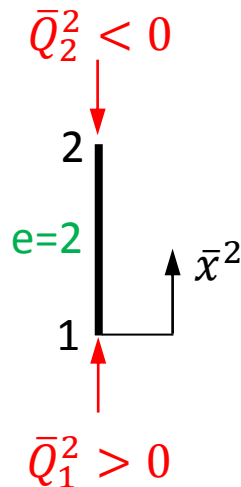
Example 4.1 (cont'd)

- As seen for each member

$$\bar{Q}_1^e = -\bar{Q}_2^e$$

i.e. **net axial force on each member is zero.**

- 1st element carries no axial force, as expected, because its both ends are fixed.
- 2nd element is in **compression** because $\bar{Q}_1^2 > 0$ (or $\bar{Q}_2^2 < 0$).
- 3rd element is in **tension** because $\bar{Q}_1^3 < 0$ (or $\bar{Q}_2^3 > 0$).



Example 4.1 (cont'd)

- **Axial stresses** in each element can be calculated.

For e=1 : $\sigma^1 = \frac{\bar{Q}_2^1}{A} = 0$

For e=2 : $\sigma^2 = \frac{\bar{Q}_2^2}{A} = -3 \frac{P}{A}$ (Negative stress indicates compression)

For e=3 : $\sigma^3 = \frac{\bar{Q}_2^3}{A} = 1.414 \frac{P}{A}$ (Positive stress indicates tension)

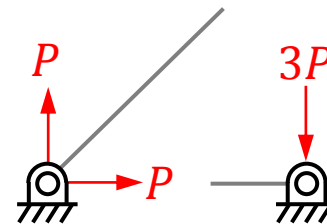
- Finally **forces at the supports** can be calculated using the 6x6 system of Slide 4-22.

At node 1 : $Q_1 = \frac{EA}{L} (1.3536\Delta_1 + 0.3536\Delta_2 - \Delta_3 - 0.3536\Delta_5 - 0.3536\Delta_6) = -P$

$Q_2 = \frac{EA}{L} (0.3536\Delta_1 + 0.3536\Delta_2 - 0.3536\Delta_5 - 0.3536\Delta_6) = -P$

At node 2 : $Q_3 = \frac{EA}{L} (-\Delta_1 + \Delta_3) = 0$

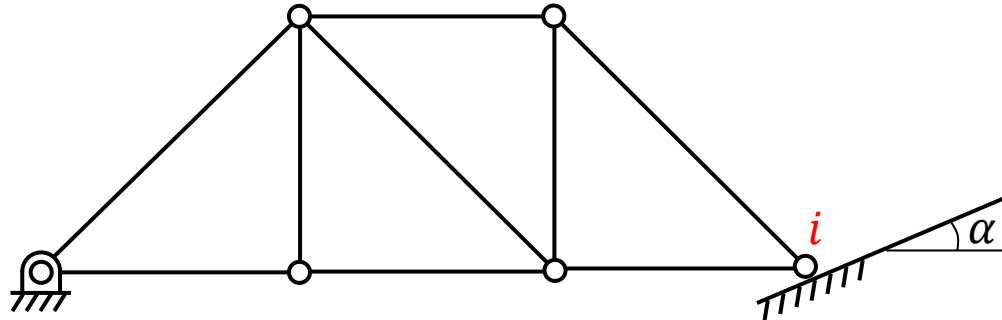
$Q_4 = \frac{EA}{L} (\Delta_4 - \Delta_6) = 3P$



Forces at the supports are the opposite of the calculated ones

Planar Truss – Constrained Motion

- Sometimes horizontal and vertical deflections of a node are not independent.
- This happens at a roller support inclined at an angle to the global xy system.



- At node i horizontal and vertical deflections are related to each other.

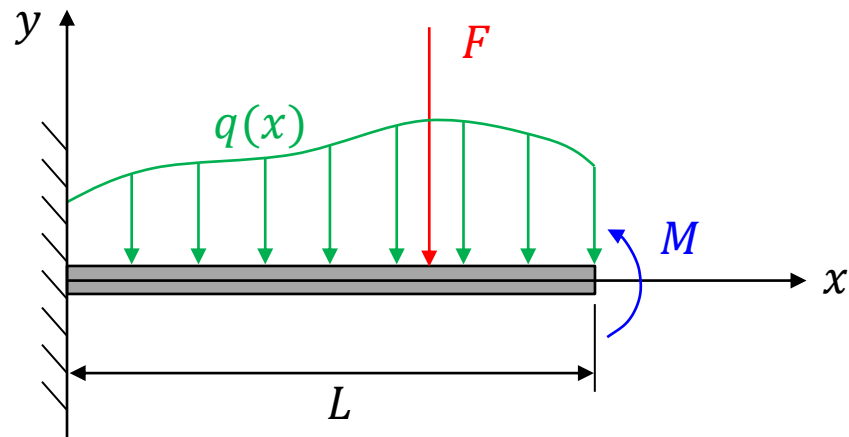
$$\Delta_{2i-1} \sin(\alpha) - \Delta_{2i} \cos(\alpha) = 0$$

Horizontal deflection of node i Vertical deflection of node i

- Details of how to handle these cases can be found in FEM textbooks.

Beam Bending

- **Beams** are long, slender structural members, generally subjected to **transverse loading** that produces significant bending effects.
- Axial deformation or twisting is not considerable for beams.

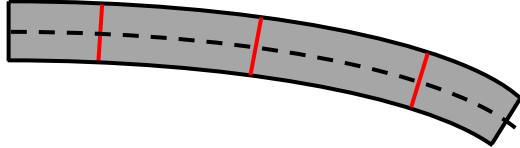


- $q(x)$ is the distributed transverse loading.
- F is a point transverse load and M is a point bending moment.
- Transverse deflection $v(x)$ is in the y direction.

Euler-Bernoulli Beam Theory

- The assumption behind the Euler-Bernoulli beam theory is that plane cross sections perpendicular to the longitudinal axis of the beam before bending, remain perpendicular to the longitudinal axis after bending.

Before bending : 

After bending : 

- Governing DE is
$$\frac{d^2}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) = q(x) \quad , \quad 0 < x < L$$

- $v(x)$: Unknown transverse deflection
- $q(x)$: Known distributed transverse load
- EI : Known flexural rigidity of the beam, i.e. product of modulus of elasticity and the second moment of inertia.

Euler-Bernoulli Beam Theory

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) = q(x) \quad , \quad 0 < x < L$$

- This is a **4th order DE**.
- IBP should be applied two times to get the weak form.
- **First IBP** gives

$$\int_{\Omega^e} -\frac{dw}{dx} \frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right) dx = \int_{\Omega^e} wq \, dx + \left[-w \frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right) \right]_{x_2^e} + \left[w \frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right) \right]_{x_1^e}$$

- **Second IBP** gives

$$\int_{\Omega^e} \frac{d^2 w}{dx^2} EI \frac{d^2 v}{dx^2} dx = \int_{\Omega^e} wq \, dx +$$
$$+ \left[-w \frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right) \right]_{x_2^e} + \left[w \frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right) \right]_{x_1^e} + \left[\frac{dw}{dx} EI \frac{d^2 v}{dx^2} \right]_{x_2^e} + \left[-\frac{dw}{dx} EI \frac{d^2 v}{dx^2} \right]_{x_1^e}$$

Euler-Bernoulli Beam Theory (cont'd)

- There are **two PVs**

Transverse deflection : v
Slope : $\frac{dv}{dx}$

- There are **two SVs**

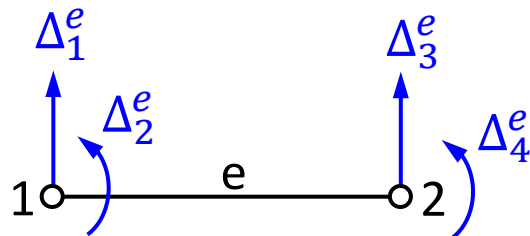
Shear force : $\frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right)$
Bending moment : $EI \frac{d^2 v}{dx^2}$

- **Sign conventions** are

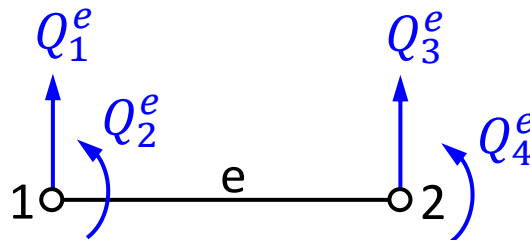
- Deflection in $+y$ direction (upward) is positive.
- CCW rotation of the beam corresponds to positive slope.
- Shear force in $+y$ direction (upward) is positive.
- CCW moment (in $+z$ direction) is positive.

Two Node Beam Element

- 2 node beam element has 4 PVs and 4 SVs



$$\Delta^e = \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} \begin{array}{l} \leftarrow \text{Transverse deflection at node 1} \\ \leftarrow \text{Slope at node 1} \\ \leftarrow \text{Transverse deflection at node 2} \\ \leftarrow \text{Slope at node 2} \end{array}$$



$$Q^e = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} \begin{array}{l} \leftarrow \text{Shear force at node 1} \\ \leftarrow \text{Bending moment at node 1} \\ \leftarrow \text{Shear force at node 2} \\ \leftarrow \text{Bending moment at node 2} \end{array}$$

Hermite Type Shape Functions

- Weak form of the problem contains 2nd derivative of the transverse deflection.
- Not only the transverse deflection, but also its first derivative, i.e. slope should be continuous, because slope is a PV too.
- Over each element FE solution is

$$v^e = \sum_{j=1}^4 S_j \Delta_j^e$$

- A C^0 continuous solution for Δ^e is NOT enough. It should be at least C^1 continuous.
- Lagrange type shape functions used previously are not suitable.
- Hermite type shape functions should be used.

Hermite Type Shape Functions (cont'd)

- Over each beam element there are 4 unknowns.
- Continuity of two variables (v and dv/dx) at two ends of an element results in 4 conditions to be satisfied.
- To satisfy these 4 conditions at least a **cubic polynomial** is necessary for v^e .

$$v^e = A + B\xi + C\xi^2 + D\xi^3$$

- Four continuity restrictions are

$$\text{At } \xi = -1 : v^e = \Delta_1^e$$

$$\text{At } \xi = -1 : \frac{dv^e}{dx} = \Delta_2^e \quad \rightarrow \quad \frac{dv^e}{d\xi} \frac{1}{J^e} = \Delta_2^e$$

$$\text{At } \xi = +1 : v^e = \Delta_3^e$$

$$\text{At } \xi = +1 : \frac{dv^e}{dx} = \Delta_4^e \quad \rightarrow \quad \frac{dv^e}{d\xi} \frac{1}{J^e} = \Delta_4^e$$

Hermite Type Shape Functions (cont'd)

- Using J^e these 4 conditions become

$$\Delta_1^e = A - B + C - D$$

$$\Delta_2^e = (B - 2C + 3D) \frac{2}{h^e}$$

$$\Delta_3^e = A + B + C + D$$

$$\Delta_4^e = (B + 2C + 3D) \frac{2}{h^e}$$

- Solve for A , B , C and D in terms of Δ_1^e , Δ_2^e , Δ_3^e and Δ_4^e .
- Substitute them into the following equation

$$\underbrace{\sum_{j=1}^4 S_j \Delta_j^e}_{v^e} = \underbrace{A + B\xi + C\xi^2 + D\xi^3}_{v^e}$$

- And identify the 4 Hermite type cubic shape functions.

Hermite Type Shape Functions (cont'd)

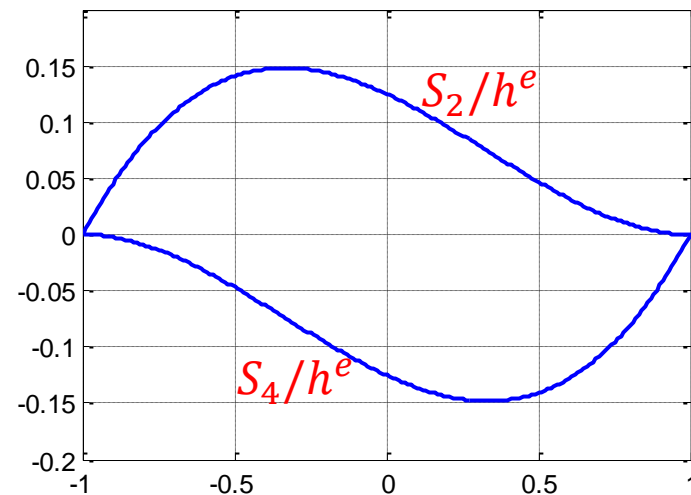
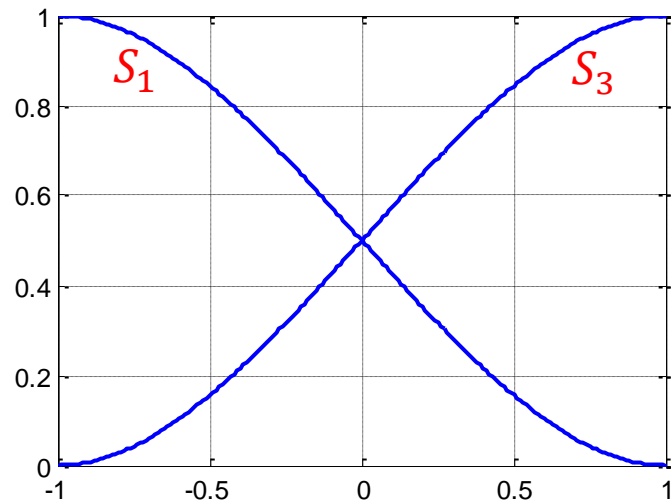
- Hermite type cubic shape functions are

$$S_1 = \frac{1}{4} (\xi^3 - 3\xi + 2)$$

$$S_2 = \frac{h^e}{8} (\xi^3 - \xi^2 - \xi + 1)$$

$$S_3 = \frac{1}{4} (-\xi^3 + 3\xi + 2)$$

$$S_4 = \frac{h^e}{8} (\xi^3 + \xi^2 - \xi - 1)$$



Elemental System for a Beam

- From Slide 4-30, elemental stiffness matrix and force vector are

$$K_{ij}^e = \int_{-1}^1 EI \frac{d^2 S_i}{d\xi^2} \left(\frac{1}{J^e}\right)^2 \frac{d^2 S_j}{d\xi^2} \left(\frac{1}{J^e}\right)^2 J^e d\xi$$

$$F_i^e = \int_{-1}^1 q S_i J^e d\xi$$

- Evaluating these using Hermite type shape functions

q is the part of the distributed transverse load, **simplified as uniform** over element e

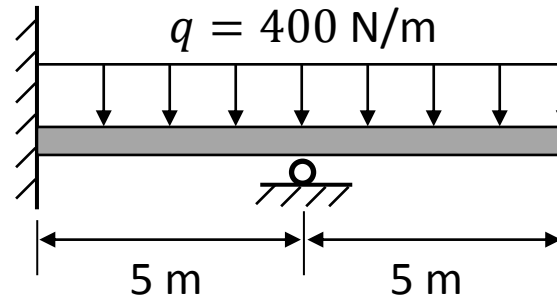
$$\underbrace{\frac{2EI}{(h^e)^3} \begin{bmatrix} 6 & 3h^e & -6 & 3h^e \\ & 2(h^e)^2 & -3h^e & (h^e)^2 \\ & & 6 & -3h^e \\ \text{sym} & & & 2(h^e)^2 \end{bmatrix}}_{K^e} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \underbrace{\frac{qh^e}{12} \begin{Bmatrix} 6 \\ h^e \\ 6 \\ -h^e \end{Bmatrix}}_{F^e} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

Example 4-2



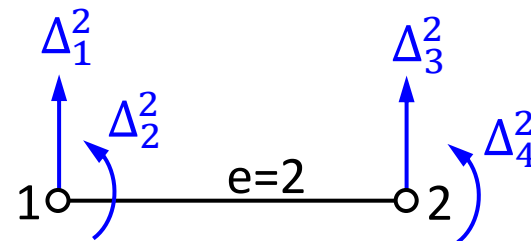
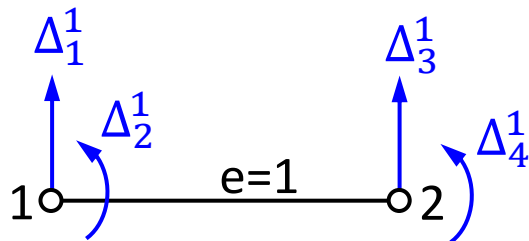
Example 4.2 : For the following clamped beam with $EI = 4 \times 10^6$ Nm, use two equal length elements to determine

- the transverse deflection of the tip
- the reaction force at the middle support.

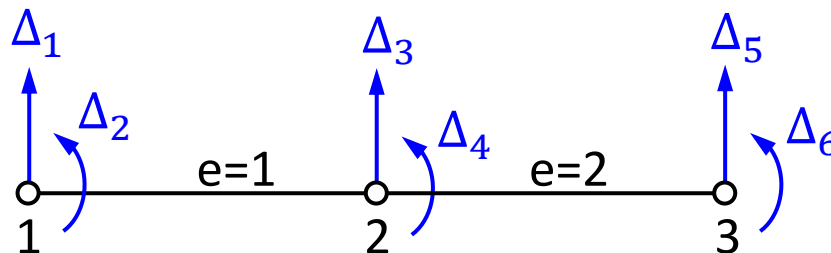


Example 4-2 (cont'd)

- EI , h^e and q are the same for both elements.
- K^e and F^e will be the same for both elements.
- There are four unknowns for each element.

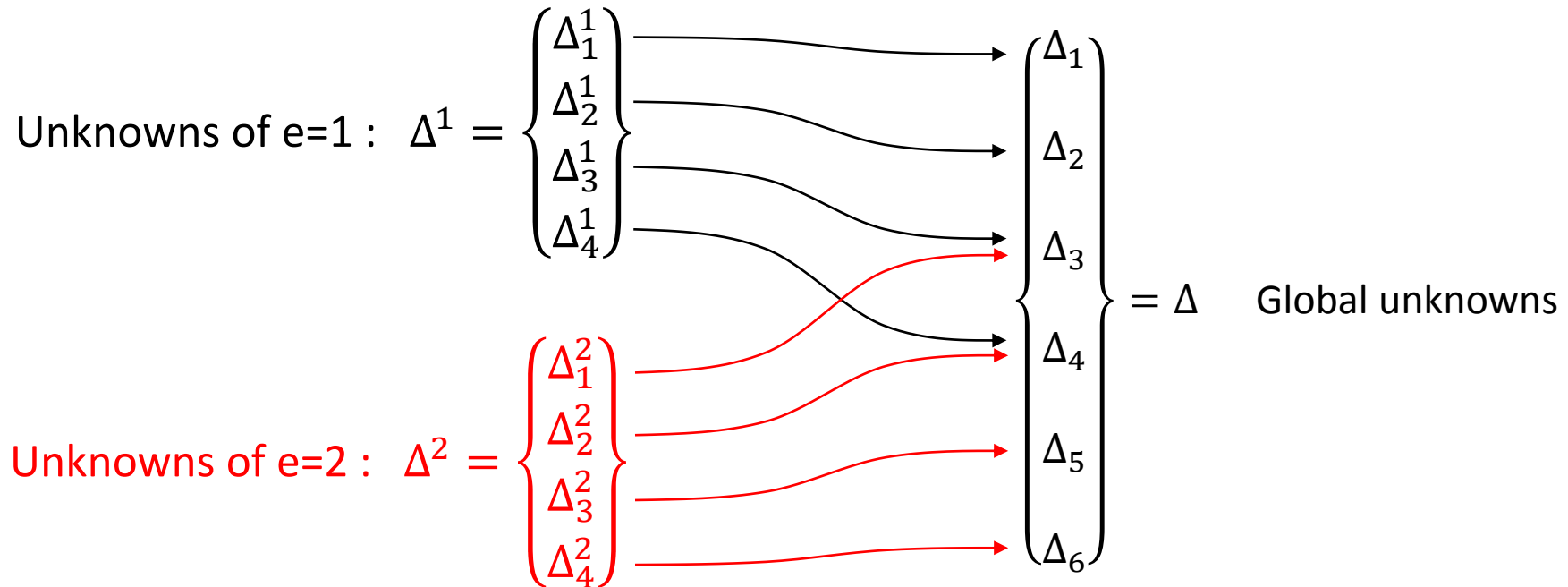


- Overall there are 3 nodes and 6 unknowns



Example 4-2 (cont'd)

- Local-to-global mapping of the unknowns are as follows



$$LtoG = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Example 4-2 (cont'd)

- Assembled system is

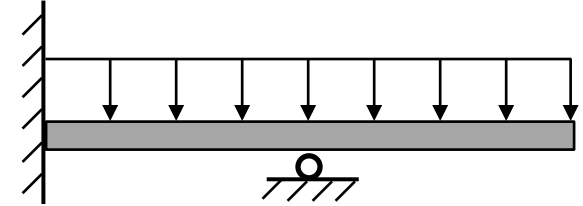
$$\frac{2EI}{(h^e)^3} \begin{bmatrix} 6 & 3h^e & -6 & 3h^e & 0 & 0 \\ 3h^e & 2(h^e)^2 & -3h^e & (h^e)^2 & 0 & 0 \\ -6 & -3h^e & 6+6 & -3h^e+3h^e & -6 & 3h^e \\ 3h^e & (h^e)^2 & -3h^e+3h^e & 2(h^e)^2+2(h^e)^2 & -3h^e & (h^e)^2 \\ 0 & 0 & -6 & -3h^e & 6 & -3h^e \\ 0 & 0 & 3h^e & (h^e)^2 & -3h^e & 2(h^e)^2 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix}$$

$$= \frac{qh^e}{12} \begin{Bmatrix} 6 \\ h^e \\ 6+6 \\ -h^e+h^e \\ 6 \\ -h^e \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1+Q_1^2 \\ Q_4^1+Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

Example 4-2 (cont'd)

- **Boundary conditions** need to be applied.

- Known deflections and slopes are **EBCs**.



- At the clamped end, transverse deflection and slope are zero.

$$\Delta_1 = 0, \quad \Delta_2 = 0$$

- At the middle support transverse deflection is zero.

$$\Delta_3 = 0$$

- Known shear forces and moments are **NBCs**.

- Middle support can not carry any bending moment

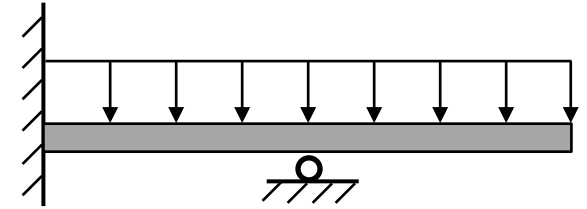
$$Q_4 = Q_4^1 + Q_2^2 = 0$$

- Free end can not carry any shear force or bending moment

$$Q_5 = 0, \quad Q_6 = 0$$

Example 4-2 (cont'd)

- Only 3 PVs ($\Delta_4, \Delta_5, \Delta_6$) are actually unknown.
- **Reduction** can be applied to the original 6x6 system.



$$\frac{2EI}{h^{e3}} \begin{bmatrix} 4h^{e2} & -3h^e & h^{e2} \\ -3h^e & 6 & -3h^e \\ h^{e2} & -3h^e & 2h^{e2} \end{bmatrix} \begin{Bmatrix} \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix} = \frac{qh^e}{12} \begin{Bmatrix} 0 \\ 6 \\ -h^e \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- Using $EI = 4 \times 10^6 \text{ Nm}$, $h^e = 5 \text{ m}$, $q = -400 \text{ N/m}$

$\{F\}$ is not changed
because $\Delta_1 = \Delta_2 = \Delta_3 = 0$

q is in $-y$ direction

- Unknown PVs are calculated as

$\Delta_4 = -0.00130 \text{ rad}$ ← Slope at the middle support. Beam rotation is CW.

$\Delta_5 = -0.01432 \text{ m}$ ← Transverse deflection at the tip. It is downward.

$\Delta_6 = -0.00339 \text{ rad}$ ← Slope at the tip. Beam rotation is CW.

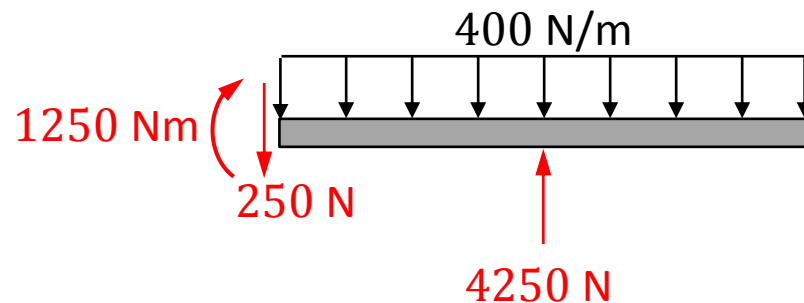
Example 4-2 (cont'd)

- To find the unknown SVs we can use the calculated PVs in the original 6x6 system.
- Unknown SVs can be calculated as

$Q_1 = -250 \text{ N}$ ← Force applied by the wall at the clamped end.

$Q_2 = -1250 \text{ Nm}$ ← Moment applied by the wall at the clamped end.

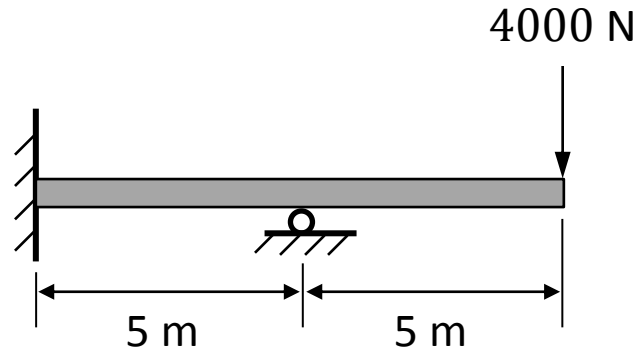
$Q_3 = 4250 \text{ N}$ ← Force acting by the middle support.



Example 4-3



Example 4.3 : Solve the same problem, but this time remove the distributed load and put a point load at the tip and put a point load at the tip

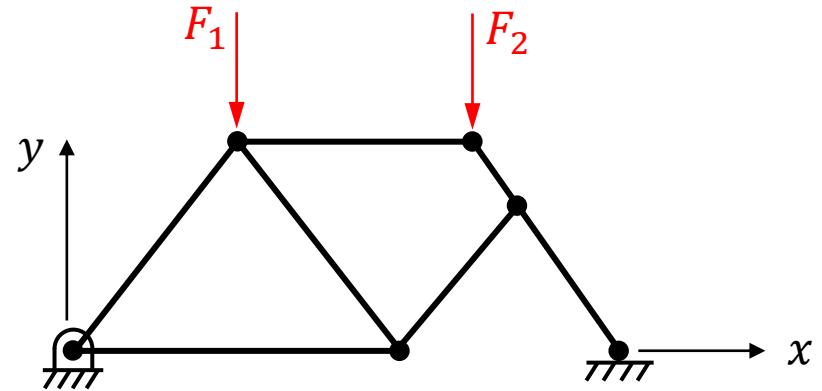
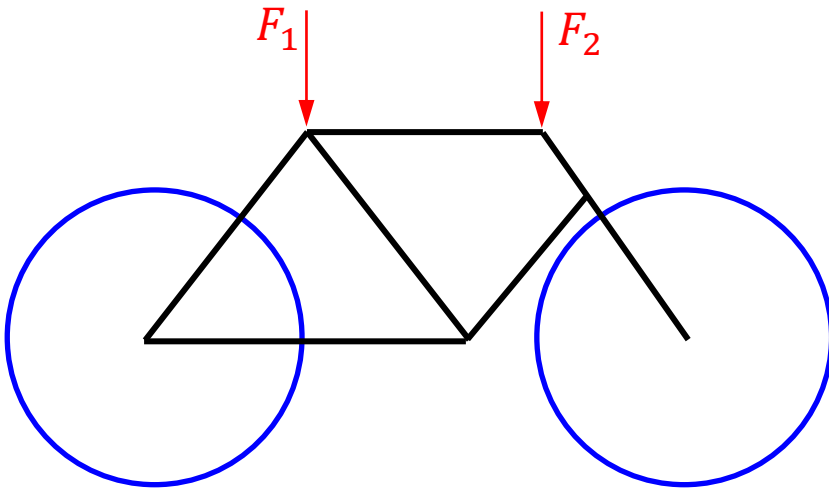


- One detail you need to pay attention is that this time

$$Q_5 = -4000$$

Planar Frames

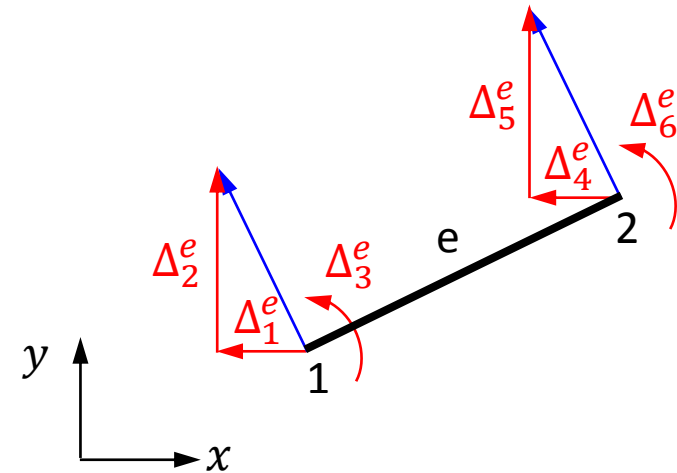
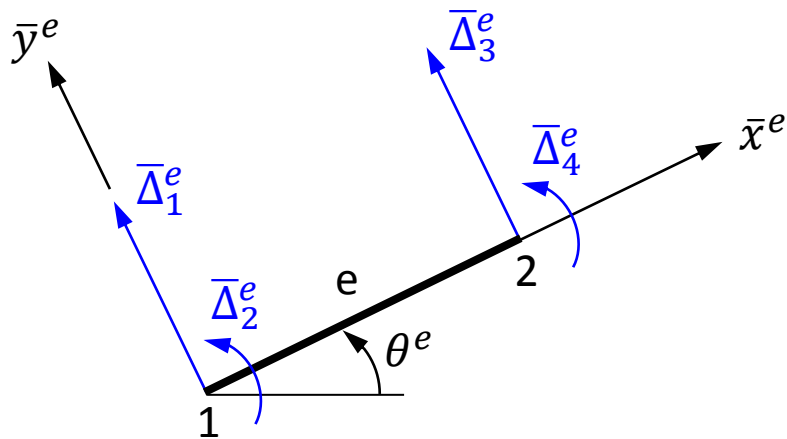
- **Frames** look like trusses, but the connections are rigid, i.e. welded or riveted.
- Each member can carry **axial force**, **shear force** and **bending moment**.



- The above bicycle frame has 7 members.
- Each member can be modeled as a single element or multiple elements.
- It is possible to think of a frame element as the superposition of truss and beam elements.

Arbitrarily Oriented Beam Element

- Frame elements are based on arbitrarily oriented beam elements.
- Similar to a truss element, it is possible to study the beam element using either the **local \bar{x}^e, \bar{y}^e coordinates** or the **global x, y coordinates**.



Unknowns in local coordinates

Node 1 : $\bar{\Delta}_1^e, \bar{\Delta}_2^e$

Node 2 : $\bar{\Delta}_3^e, \bar{\Delta}_4^e$

Unknowns in global coordinates

Node 1 : $\Delta_1^e, \Delta_2^e, \Delta_3^e$

Node 2 : $\Delta_4^e, \Delta_5^e, \Delta_6^e$

Transformation Matrix of a Beam Element

- Relation between local and global unknowns are

$$\bar{\Delta}_1^e = -\sin(\theta^e) \Delta_1^e + \cos(\theta^e) \Delta_2^e$$

$$\bar{\Delta}_2^e = \Delta_3^e$$

$$\bar{\Delta}_3^e = -\sin(\theta^e) \Delta_4^e + \cos(\theta^e) \Delta_5^e$$

$$\bar{\Delta}_4^e = \Delta_6^e$$

- These relations can be expressed using the following transformation matrix.

$$\begin{Bmatrix} \bar{\Delta}_1^e \\ \bar{\Delta}_2^e \\ \bar{\Delta}_3^e \\ \bar{\Delta}_4^e \end{Bmatrix} = \begin{bmatrix} -\beta & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \Delta_5^e \\ \Delta_6^e \end{Bmatrix}$$

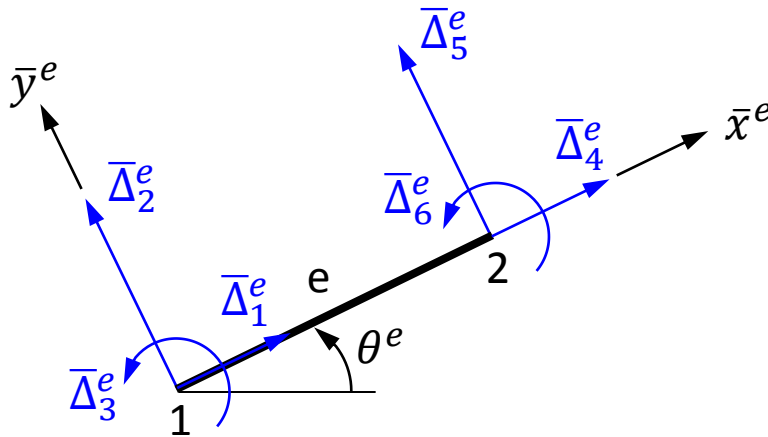
where $\alpha = \cos(\theta^e)$ and $\beta = \sin(\theta^e)$

Transformation matrix of the beam element

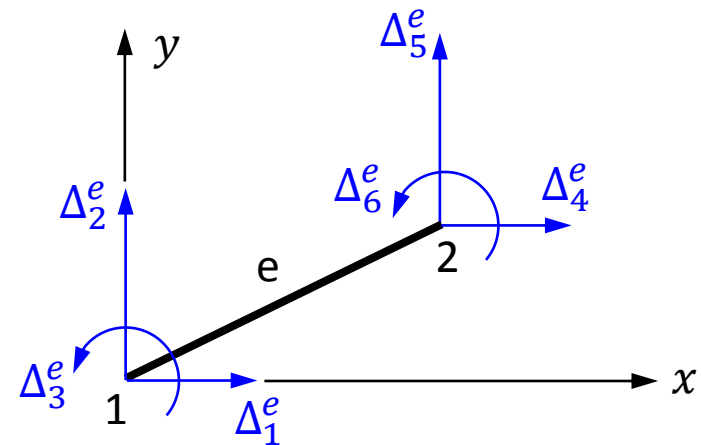
Frame Element

- We now have transformation matrices for arbitrarily oriented beam and truss elements.
- Frame elements carry axial force, shear force and bending moment.
- They can be obtained by the **superposition of beam and truss elements**.
- Frame element has 3 unknowns at each node.

Frame element in local coordinates



Frame element in global coordinates



Frame Element (cont'd)

- Elemental system of the frame element in local unknowns is obtained by the proper combination of those of truss and beam elements

$$[\bar{K}^e]\{\bar{\Delta}^e\} = \{F^e\}$$

q is the constant distributed transverse load

These are coming from $[\bar{K}^e]$ of the truss element (Slide 4-9)

The rest is coming from $[\bar{K}^e]$ of the beam element (Slide 4-37)

$$\underbrace{\begin{bmatrix}
 \frac{EA}{h^e} & 0 & 0 & -\frac{EA}{h^e} & 0 & 0 \\
 \frac{12EI}{(h^e)^3} & \frac{6EI}{(h^e)^2} & 0 & -\frac{12EI}{(h^e)^3} & \frac{6EI}{(h^e)^2} & 0 \\
 0 & \frac{4EI}{h^e} & 0 & 0 & -\frac{6EI}{(h^e)^2} & \frac{2EI}{h^e} \\
 \text{Symmetric} & & & \frac{EA}{h^e} & 0 & 0 \\
 0 & 0 & 0 & 0 & \frac{12EI}{(h^e)^3} & -\frac{6EI}{(h^e)^2} \\
 0 & 0 & 0 & 0 & \frac{4EI}{h^e} & \frac{2EI}{h^e}
 \end{bmatrix}}_{[\bar{K}^e]} \underbrace{\begin{Bmatrix} \bar{\Delta}_1^e \\ \bar{\Delta}_2^e \\ \bar{\Delta}_3^e \\ \bar{\Delta}_4^e \\ \bar{\Delta}_5^e \\ \bar{\Delta}_6^e \end{Bmatrix}}_{\{\bar{\Delta}^e\}} = \underbrace{\begin{Bmatrix} 0 \\ \frac{qh^e}{2} \\ \frac{q(h^e)^2}{12} \\ 0 \\ \frac{qh^e}{2} \\ -\frac{q(h^e)^2}{12} \end{Bmatrix}}_{\{F^e\}}$$

Frame Element (cont'd)

- Similarly transformation matrix of the frame element is obtained by the proper combination of those of truss and beam elements

$$\begin{Bmatrix} \bar{\Delta}_1^e \\ \bar{\Delta}_2^e \\ \bar{\Delta}_3^e \\ \bar{\Delta}_4^e \\ \bar{\Delta}_5^e \\ \bar{\Delta}_6^e \end{Bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & -\beta & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[T^e]} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \Delta_5^e \\ \Delta_6^e \end{Bmatrix}$$

These 2 eqns are coming from the transformation matrix of the truss element (Slide 4-8)

Remaining 4 eqns are coming from the transformation matrix of the beam element (Slide 4-48)

- Elemental system in global unknowns is obtained as

$$[K^e] \{\Delta^e\} = \{F^e\}$$

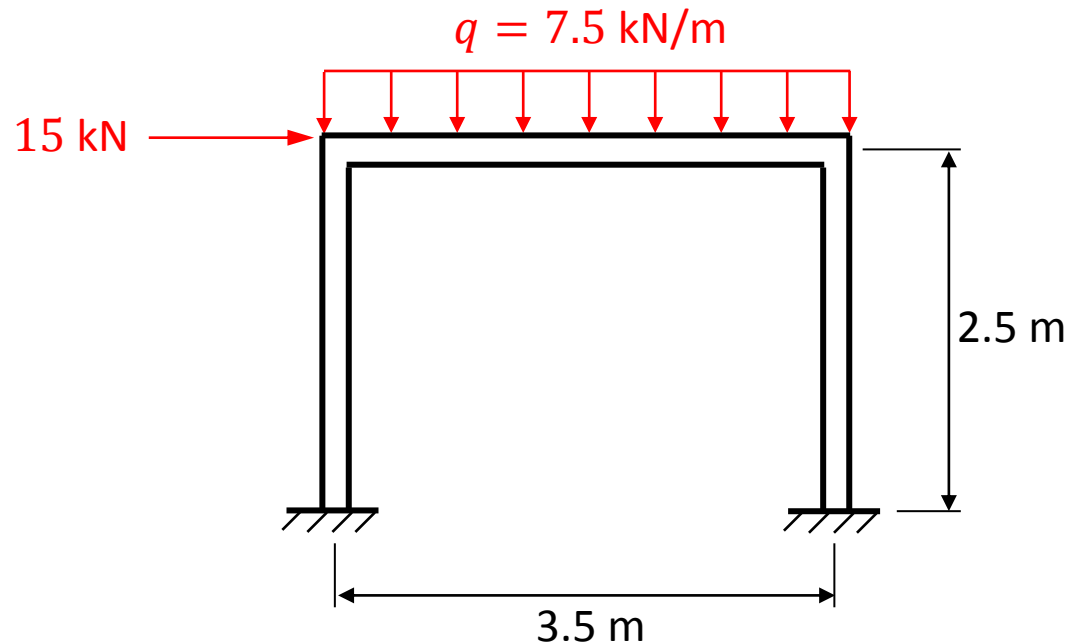
$$[K^e] = [T^e]^T [\bar{K}^e] [T^e] \qquad \{F^e\} = [T^e]^T \{\bar{F}^e\}$$

Example 4-3



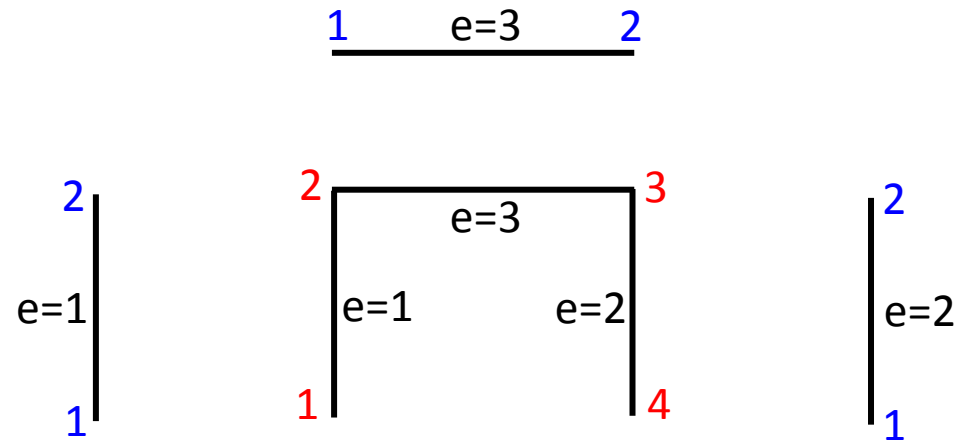
Example 4.3 : Using three elements, determine the deflections and rotations at the joints of the following frame. Draw bending moment and shear force diagrams for all elements. Calculate the reactions at the supports.

For all members $E = 200 \text{ GPa}$, $I = 2.7 \times 10^{-6} \text{ m}^4$, $A = 4.4 \times 10^{-4} \text{ m}^2$



Example 4-3 (cont'd)

- Element and global/local node numbering are shown below.



- Orientation** of the elements are

$$\theta^1 = \pi/2 , \theta^2 = \pi/2 , \theta^3 = 0$$

- Element lengths** are

$$h^1 = 2.5 , h^2 = 2.5 , h^3 = 3.5$$

- E , I and A** are the same for all elements.

Example 4-3 (cont'd)

- Using Slide 4-50 calculate $[\bar{K}^e]$ and $\{\bar{F}^e\}$ for each element

- For e=1: $[\bar{K}^1] = 10^6 \begin{bmatrix} 35.2000 & 0 & 0 & -35.2000 & 0 & 0 \\ 0 & 0.4147 & 0.5184 & 0 & -0.4147 & 0.5184 \\ 0 & 0.5184 & 0.8640 & 0 & -0.5184 & 0.4320 \\ -35.2000 & 0 & 0 & 35.2000 & 0 & 0 \\ 0 & -0.4147 & -0.5184 & 0 & 0.4147 & -0.5184 \\ 0 & 0.5184 & 0.4320 & 0 & -0.5184 & 0.8640 \end{bmatrix}$

$$\{\bar{F}^1\} = \{0 \ 0 \ 0 \ 0 \ 0 \ 0\}^T \quad (\text{all zero because } q = 0 \text{ for } e=1)$$

- For e=2: $[\bar{K}^2] = [\bar{K}^1]$ (because E, A, I and h are the same for both elements)

$$\{\bar{F}^2\} = \{\bar{F}^1\} \quad (\text{because } q = 0 \text{ for } e=2, \text{ too})$$

Example 4-3 (cont'd)

- For $e=3$: $[\bar{K}^3] = 10^6 \begin{bmatrix} 25.1429 & 0 & 0 & -25.1429 & 0 & 0 \\ 0 & 0.1511 & 0.2645 & 0 & -0.1511 & 0.2645 \\ 0 & 0.2645 & 0.6171 & 0 & -0.2645 & 0.3086 \\ -25.1429 & 0 & 0 & 25.1429 & 0 & 0 \\ 0 & -0.1511 & -0.2645 & 0 & 0.1511 & -0.2645 \\ 0 & 0.2645 & 0.3086 & 0 & -0.2645 & 0.6171 \end{bmatrix}$

$$\{\bar{F}^3\} = 10^4 \begin{pmatrix} 0 \\ -1.3125 \\ -0.7656 \\ 0 \\ -1.3125 \\ 0.7656 \end{pmatrix}$$

Example 4-3 (cont'd)

- Now the transformation matrices for each element need to be calculated using Slide 4-51.

- For e=1: $[T^1] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

- For e=2: $[T^2] = [T^1]$ (because $\theta^2 = \theta^1$)

- For e=3: $[T^3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (This is the unity matrix because $\bar{x}^3 = x$)

Example 4-3 (cont'd)

- $[K^e]$ and $\{F^e\}$ of each element can be calculated using

$$[K^e] = [T^e]^T [\bar{K}^e] [T^e] \quad \text{and} \quad \{F^e\} = [T^e]^T \{\bar{F}^e\}$$

- For e=1: $[K^1] = 10^6 \begin{bmatrix} 0.4147 & 0.0000 & -0.5184 & -0.4147 & -0.0000 & -0.5184 \\ 0.0000 & 35.2000 & 0.0000 & -0.0000 & -35.2000 & 0.0000 \\ -0.5184 & 0.0000 & 0.8640 & 0.5184 & -0.0000 & 0.4320 \\ -0.4147 & -0.0000 & 0.5184 & 0.4147 & 0.0000 & 0.5184 \\ -0.0000 & -35.2000 & -0.0000 & 0.0000 & 35.2000 & -0.0000 \\ -0.5184 & 0.0000 & 0.4320 & 0.5184 & -0.0000 & 0.8640 \end{bmatrix}$

$$\{F^1\} = \{0 \ 0 \ 0 \ 0 \ 0 \ 0\}$$

- For e=2: $[K^2] = [K^1] \quad , \quad \{F^2\} = \{F^1\}$

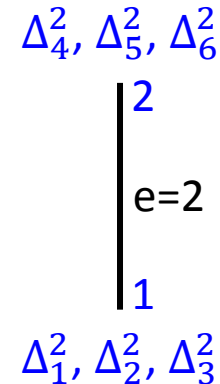
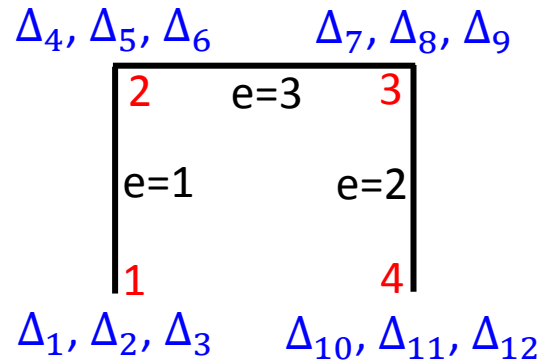
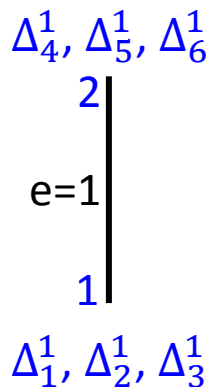
- For e=3: $[K^3] = [\bar{K}^3] \quad , \quad \{F^3\} = \{\bar{F}^3\}$

Example 4-3 (cont'd)

- Now we have three 6x6 elemental systems.
- Let's assemble them into the 12x12 global system.
- LtoG mapping is as follows

$$LtoG = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 10 & 11 & 12 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

$$\Delta_1^3, \Delta_2^3, \Delta_3^3 \quad \begin{array}{c} \text{1} \\ \hline \text{e=3} \\ \hline \text{2} \end{array} \quad \Delta_4^3, \Delta_5^3, \Delta_6^3$$



Example 4-3 (cont'd)

- Assembled global system is 12x12.
- At nodes 1 and 4 all three unknowns are known and they are zero.

$$\Delta_1 = \Delta_2 = \Delta_3 = 0 \quad , \quad \Delta_{10} = \Delta_{11} = \Delta_{12} = 0$$

$$10^6 \begin{bmatrix} 0.5184 & -0.0000 & 0.4320 & 0 & 0 & 0 \\ 25.5576 & 0.0000 & 0.5184 & -25.1429 & 0 & 0 \\ 0.0000 & 35.3511 & 0.2645 & 0 & -0.1511 & 0.2645 \\ 0.5184 & 0.2645 & 1.4811 & 0 & -0.2645 & 0.3086 \\ -25.1429 & 0 & 0 & 25.5576 & 0.0000 & 0.5184 \\ 0 & -0.1511 & -0.2645 & 0.0000 & 35.3511 & -0.2645 \\ 0 & 0.2645 & 0.3086 & 0.5184 & -0.2645 & 1.4811 \end{bmatrix} \begin{Bmatrix} \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \\ \Delta_9 \end{Bmatrix}$$

Q_4 is the given horizontal point load at node 2

Q_5 and Q_6 are zero because there is no vertical point load or point moment at node 2

Q_7 , Q_8 and Q_9 are zero because there is no horizontal or vertical point load or point moment at node 3.

$$= 10^4 \begin{Bmatrix} 0 \\ -1.3125 \\ -0.7656 \\ 0 \\ -1.3125 \\ 0.7656 \end{Bmatrix} + \begin{Bmatrix} 15000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example 4-3 (cont'd)

- Solving for the unknown deflections we get

$$\begin{Bmatrix} \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \\ \Delta_9 \end{Bmatrix} = \begin{Bmatrix} 0.02863 \\ -0.00024 \\ -0.01489 \\ 0.02820 \\ -0.00049 \\ -0.00164 \end{Bmatrix}$$

} Horizontal deflection, vertical deflection and rotation of **node 2**

} Horizontal deflection, vertical deflection and rotation of **node 3**

- Both nodes 2 and 3 move in $+x$ and $-y$ directions. Also they rotate CW.
- Now the forces and moments at the supports ($Q_1, Q_2, Q_3, Q_{10}, Q_{11}, Q_{12}$) can be calculated.
- Also axial stress, shear stress and bending stress over each element can be determined.
- Question : Will the solution improve by using more elements?